Aeroelasticity of Structurally Nonlinear Lifting Surfaces Using Linear Modally Reduced Aerodynamic Generalized Forces

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Structural nonlinear effects in aeroelasticity can be important and affect overall nonlinear behavior in cases where the aerodynamic forces are still linear. Such is the case of the joined-wing configuration in attached flow. Since modal reduction of the structural deformation problem is often not successful when complex lifting surface configurations are involved, full order finite element analysis is used to model structural behavior here, static or dynamic. Linear unsteady aerodynamics of lifting surfaces in aeroelasticity is usually modeled in the frequency domain using modal coordinates. The goal of the present effort is to combine the full order nonlinear finite element analysis of the structure with modally based generalized unsteady aerodynamic forces. Initial studies in which coupling of full order nonlinear finite element models with generalized aerodynamic force models is compared to coupling of full order nonlinear finite element models with full order aerodynamic panel methods is presented here with focus on the steady case. Unsteady cases will be described in a future paper.

Nomenclature

\(\alpha\) Angle of attack
\(V_\infty\) Freestream velocity
\(\rho_\infty\) Air density
\(\Gamma\) Circulation
\(x, y, z\) Coordinate system
\(L_{\text{ref}}\) Reference aerodynamic load
\(P_{\text{ref}}\) Reference non-aerodynamic load
\(N_{\text{step}}\) Number of load steps
\(\lambda\) Generic load step
\(n\) Generic iteration
\(N\) Number of panels
\(k\) Reduced frequency
\(a\) Geometrical dimension (different for each analyzed case)
\(E\) Elastic modulus
\(\rho\) Material density
\(v\) Poisson's ratio
\(h\) Thickness of the plate
\(w\) Displacement in the \(z\) direction
\(q\) Dynamic pressure
\(\mathcal{R}\) Number of shape vectors
\(Z_{\text{loc}}^S\) Local \(z\) coordinate of the \(i^{\text{th}}\) structural node on surface \(S\)
\(K_G^e\) Geometric stiffness matrix at element level
\(K_G\) Geometric stiffness matrix at structural level

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\( K_E \) Elastic stiffness matrix at structural level

\( \omega \) Small rigid rotation vector

\( i \) Unit vector \((x\) direction\)

\( j \) Unit vector \((y\) direction\)

\( k \) Unit vector \((z\) direction\)

\( \Gamma \) Vector which contains the vorticity of all panels

\( A^\Gamma \) Matrix of the influence coefficients

\( H_{\text{RHS}} \) Right hand side vector used in the imposition of the Wall Tangency Condition

\( L \) Vector containing all lifting forces

\( I_\Gamma \) Transformation matrix

\( u^S_i \) Translational displacements (no rotations) of the structural nodes on surface \( S \)

\( u^d_i \) Translational displacements (no rotations) of all the structural nodes

\( u \) Displacements of all structural nodes

\( x \) Vector containing the global coordinates of all structural nodes

\( x^{\text{start}} \) Vector containing the global coordinates of all structural nodes (undeformed configuration)

\( X^S \) Vector containing the updated coordinates of the structural nodes on surface \( S \)

\( Z^S_{\text{loc}} \) Vector containing the coordinates \( Z^S_{\text{loc}} \)

\( Z^S \) Vector containing the local \( z \) coordinates of the control points on surface \( S \)

\( A \) Constant vector used in the definition of the aerodynamic loads

\( B \) Constant matrix used in the definition of the aerodynamic loads

\( C \) Aerodynamic matrix used in the creation of the aerodynamic tangent matrix

\( C_Z \) Approximated \( C \)

\( L_{\text{str}} \) Aerodynamic loads applied to the structural mesh

\( P_{\text{str}} \) Current non-aerodynamic loads

\( P_{\text{ext}} \) Applied non-aerodynamic loads

\( F_{\text{int}} \) Internal forces

\( P_{\text{unb}} \) Unbalanced loads

\( K_T \) Structural tangent matrix

\( A_0 \) Generalized aerodynamic matrix (steady case)

\( A_Z \) Generalized aerodynamic tangent matrix (steady case)

\( \Psi \) Basis of known shape vectors used to reduce the system

\( \Phi_i \) Generic shape vector used in the creation of the basis \( \Psi \)

\( q \) Generalized coordinates

\( Q \) Generalized forces

\( K_{\text{aero}} \) Aerodynamic tangent matrix

\( K_{T\text{aero}} \) Approximated aerodynamic tangent matrix

\( K_{\text{Tangent}} \) Tangent matrix

\( \text{Subscript} \)

\( \text{IP} \) In-plane

\( \text{OUT} \) Out-of-plane

\( \text{Superscript} \)

\( S \) Referred to the surface \( S \)

\( T \) The transpose is calculated

\( -1 \) The inverse of the matrix is calculated

\( \text{step} \lambda \text{iter} \text{n} \) Referred to step \( \lambda \) and iteration \( n \)

\( \text{Acronym} \)

\( \text{JW} \) Joined Wing

\( \text{WTC} \) Wall Tangency Condition

\( \text{IPS} \) Infinite Plate Spline
THE Joined Wing (JW) configuration has been the subject of aerodynamic, structural, and design optimization studies for almost 30 years. Its aeroelastic behavior, however, and its effect on design are still not completely understood, and only a very small number of exploratory studies addressed JW aeroelasticity in a satisfactory way.

With Joined-Wings, the substantial tail-rear wing surface can be under significant in-plane compression and the inboard part of the wing is under in-plane tension. Structural geometric nonlinearity becomes important and can affect aeroelastic behavior by leading to divergence of the rear wing, changing natural frequencies under maneuver conditions with the resulting impact on flutter, or leading to interactions between static and dynamic instabilities. Aeroelastic investigation of geometrically nonlinear lifting surfaces in the past few years covers high-aspect ratio wings of high-altitude long-endurance aircraft (HALE), strut-braced wings, wind tunnel models of delta and beam-like wings, and joined-wing configurations. With progress in the area of integrated Structural Finite Element (FE) / Computational Fluid Dynamics (CFD) of flight vehicles, the natural tendency, when joined-wing aeroelasticity is considered, is to model the problem using emerging integrated FE/CFD capabilities. That will allow nonlinearities to be captured by the modeling whether their source is structural or aeroelastic. The resulting simulations, however, are computationally demanding, and model preparation for such FE/CFD simulation can be time consuming too due to CFD mesh generation. Important aspects of the aeroelasticity of joined wings and other flight vehicle configurations are controlled, however, by structural nonlinear behavior, while the associated aerodynamic behavior remains linear. Such linear aerodynamic behavior can be modeled by linear panel methods, and, in the case of thin lifting surfaces, by linear lifting surface methods. And, as is the case with linear aeroelasticity of flight vehicles, full order structural finite element models can be coupled with full order aerodynamic panel methods for the static case, while in the unsteady case, modally reduced models for both structures and aerodynamics are coupled. Modally reduced models have been demonstrated to perform successfully even in the static linear case. Coupling of full order FE models with time marching unsteady aerodynamic panel models has been demonstrated in the unsteady case too. But the configurations involved were limited to simple configurations in incompressible flow, and any extension of similar unsteady panel time domain capabilities to the case of complex 3D configurations is not trivial. With commercial codes available, such as the Doublet Lattice Method codes, or the ZAERO codes for modeling linear unsteady aerodynamics on general 3D configurations, a method that would allow coupling of the generalized aerodynamic forces obtained by such codes with full order finite element structural models would be beneficial. We present here a modeling capability for JW aeroelastic behavior affected by nonlinear structural behavior and linear aerodynamic forces given in the form of generalized forces in modal coordinates. To assess the accuracy of transformation of aerodynamic effects from generalized modal coordinates to full-order finite element coordinates, we focus here on the steady case, and a Vortex Lattice technique is used for aerodynamic simulation. This will allow comparison of results obtained with the full order vortex lattice panel model and the aerodynamic model based on generalized modal coordinates. The structural part of the capability consists of a dedicated nonlinear finite element code for 3D plate assembly configurations.

II. Nonlinear Structural Model

The geometrically nonlinear structural model is built using flat triangular elements. The tangent stiffness matrix is built adding the linear elastic stiffness matrix and the geometric stiffness matrix. The geometric stiffness matrix is built applying the load perturbation method: the gradient (with respect to the coordinates) of the nodal force vector (when the stresses are considered fixed) is calculated. The geometric stiffness matrix is calculated adding the linear elastic stiffness matrix and the geometric stiffness matrix. The tangent matrix is calculated adding 4 matrices:

\[
{[K_e}^{TOTAL}]_G = {[K_e}^{mem}]_G + {[K_e}^{plate}]_G + {[K_e}^{mem}]_G + {[K_e}^{plate}]_G.
\] (1)

The matrix \([K_e]^{mem}_G\), representing the in-plane contribution of the plane stress triangular element (CST), is obtained taking the gradient of the nodal forces. The matrix \([K_e]^{plate}_G\), representing the in-plane contribution of the flat triangular plate bending element, is calculated using a similar approach applied to the triangular element based on the Discrete Kirchoff Theory (DKT). The matrix \([K_e]^{mem}_G\) representing the out-of-plane contribution of the membrane, is calculated considering the change of a vector force which is subjected to a small rigid rotation vector \(\omega\). Similar approach is conducted in order to calculate the matrix

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\[ [K_{G_{\text{OUT}}}^{c}] \] which represents the out-of-plane contribution of the plate.
A particular procedure\textsuperscript{27} is then used in order to remove the rigid body motion and calculate the unbalanced load as the analysis (Newton Raphson) progresses.

III. Vortex Lattice Formulation

The geometry of a generic non-planar wing system is reported in figure\textsuperscript{1} The velocity \( V_{\infty} \) is assumed directed along +\( x \). The wing is discretized using wing segments (called also wing surfaces in the paper). The Wall Tangency Condition (WTC) has to be imposed for all panels of all surfaces (the number of panels is \( N \)). The consequent equation is:

\[ A^T \cdot \Gamma = -V_{\infty} H_{\text{RHS}} \]  

(2)

where

\[ H_{\text{RHS}} = \begin{bmatrix} i^T \cdot n_1 & i^T \cdot n_2 & i^T \cdot n_3 & \ldots & i^T \cdot n_N \end{bmatrix}^T \]  

(3)

Figure 1. A generic non-planar wing system.

Figure 2. Surface \( j \). Geometry and notation.
\( A^F \) is the matrix which contains the influence coefficients and \( \Gamma \) is a vector which contains all the circulations \( \Gamma_1, \Gamma_2...\Gamma_N \). \( i \) is the unit vector (x direction) and \( n_1, n_2, n_3 ... n_N \) are the normals to the panels 1, 2, 3, ..., \( N \) respectively. The normals are calculated at the control points of the panels. The lifting forces can be calculated for all aerodynamic panels (notice that the component in x direction is always zero because the aerodynamic force is perpendicular to the flow speed) and can be rearranged in a vector as follows:

\[
L = \rho_\infty V_\infty I_\Gamma \cdot \Gamma = -\rho_\infty V_\infty^2 I_\Gamma \cdot \left[A^F\right]^{-1} H_{\text{RHS}}
\]

where the transformation matrix \( I_\Gamma \) depends on the orientation of the panels in space.

### IV. Calculation of the Aerodynamic Forces

The vortices used in the vortex lattice formulation are not moved even if the structure deforms. This assumption is valid if the displacements are not too large. However, even if the vortices are kept fixed, their intensity changes because the Wall Tangency Condition has to be imposed using the new orientation of the elements in the space. Relations (2) and (3) are still valid, but it has to be clear that the matrix \( A^F \) does not change and the matrix \( I_\Gamma \) does not change as well (this is because the aerodynamic force direction depends on the external product between the vectors that identify the vortices and the vortices do not move under the hypothesis of linear aerodynamic theory). The vorticity vector \( \Gamma \) does change during the deformation process and reason is that the Wall Tangency Condition has to imposed considering the new directions of the normals to the aerodynamic panels at the control points. Thus, the vectors \( H_{\text{RHS}} \) and \( \Gamma \) do change even if the aerodynamic formulation is linear. The structure includes geometric nonlinearities. Therefore, the aerodynamic pressure can not be applied at once: load steps have to be defined. For example, if \( N_{\text{step}} \) is the number of steps that are considered (it has to be relatively large in order to achieve the convergence), the reference aerodynamic load is:

\[
L_{\text{ref}} = -\frac{\rho_\infty V_\infty^2}{N_{\text{step}}}
\]

At the end of the generic iteration \( n \) corresponding to the load step \( \lambda \), the aerodynamic loads (applied at the load points of the aerodynamic mesh) will be

\[
L^{\text{step } \lambda \text{ iter } n} = \lambda L_{\text{ref}} I_\Gamma \left[A^F\right]^{-1} H_{\text{RHS}}^{\text{step } \lambda \text{ iter } n}
\]

The vector \( H_{\text{RHS}}^{\text{step } \lambda \text{ iter } n} \) depends on the initial coordinates of the control points at the iteration \( n \) and, also, it has a contribution from the displacements referred to the coordinates at the beginning of the iteration \( n \). The matrix that multiplies the displacements has the meaning of aerodynamic tangent matrix.

In order to impose the Wall Tangency Condition, the slopes of the structural shape have to be calculated. However, the shape is known only at the points of the structural mesh and the derivatives calculated at the control points are not known. Thus, an interpolation is necessary and the Infinite Plate Spline method is applied. The method is applied for each wing segment into which the wing is divided (notice that the wing segments are not the aerodynamic panels, but only macro elements in which the wing is divided). The undeformed initial configuration will represent the reference plane used in the IPS approximation. In general the wing system can be non-planar and this approach is general and can be used to analyze wings generally positioned in the space (such as Joined Wings). Notice that different wing segments have different reference planes.

\( u^{\text{step } \lambda \text{ iter } n} \) is the displacement vector referred to the coordinate vector \( x^{\text{step } \lambda \text{ iter } n} \) of the beginning of the \( n^{\text{th}} \) iteration (Updated Lagrangian Formulation). Under the assumption that the displacements are not large, it is possible to consider that the projections of the points in the deformed positions on the reference planes do not change. This leads to the writing of transformations that do not change during the iterative process (Newton Raphson). The used procedure adopted to obtain the “exact” aerodynamic tangent matrix \( K^{\text{step } \lambda}_{\text{Taeo}} = -\lambda L_{\text{ref}} C \) is summarized below (more details can be found in a previous authors’ work6).

The displacement vector (rotations not included) \( u^{\text{step } \lambda \text{ iter } n}_S \) of all structural nodes included in the generic wing segment \( S \) (see Figures 1 and 2) is calculated as the product between a constant transformation matrix and the displacement vector \( u^{\text{step } \lambda \text{ iter } n} \). The global coordinates \( x^{\text{step } \lambda \text{ iter } n}_S \) at the beginning of
the $n$th iteration of all structural nodes included in the generic wing segment $S$ are calculated as a product between a constant transformation matrix and the vector $x_{\text{step}\lambda \text{iter }n}$. The global coordinates of all nodes included in wing segment $S$ at the end of the $n$th iteration are calculated by adding the two contributions:

$$X_{\text{step}\lambda \text{iter }n}^S = x_{\text{step}\lambda \text{iter }n}^S + u_d^S_{\text{step}\lambda \text{iter }n}$$  \hspace{1cm} (7)

Considering the above mentioned transformation matrices, $X_{\text{step}\lambda \text{iter }n}^S$ is a linear function of $u_{\text{step}\lambda \text{iter }n}$ and $x_{\text{step}\lambda \text{iter }n}$.

A local coordinate system (see Figure 3), located on the generic wing segment $S$, is used and the vector of global coordinates $X_{\text{step}\lambda \text{iter }n}^S$ is transformed into a vector $X_{\text{loc}\lambda \text{iter }n}^S$ which contains the local coordinates of all nodes included in wing segment $S$. $X_{\text{loc}\lambda \text{iter }n}^S$ is also a linear function of the displacements and coordinates at the beginning of the iteration $n$. From the vector $X_{\text{loc}\lambda \text{iter }n}^S$ it is possible to extract all components perpendicular to the surface by multiplying that vector by a constant matrix. $Z_{\text{loc}\lambda \text{iter }n}^S$ is the result of this operation and as the original vector is a function of the displacement and coordinates vectors ($u_{\text{step}\lambda \text{iter }n}$ and $x_{\text{step}\lambda \text{iter }n}$) respectively.

The theory of Infinite Plate Spline (IPS) is used and the slope (deformed configuration) is calculated at the control points of the aerodynamic panels of the generic wing segment $S$ (this operation is of course done for all wing segments). The local angle of attack is a function of the control points of the aerodynamic panels of the generic wing segment $S$. Using IPS theory it is not difficult to show that $Z_{\text{loc}\lambda \text{iter }n}^S$ is a function of $u_{\text{step}\lambda \text{iter }n}$ and $x_{\text{step}\lambda \text{iter }n}$ as well. $H_{\text{RHS}}^S$ is calculated putting all values, relative to the control points of all panels in a vector (equation 3 is applied to the panels on surface $S$).

$$H_{\text{RHS}}^S_{\text{step}\lambda \text{iter }n} = \mathbf{H}_{\text{RHS}}^S - \frac{dZ_{\text{loc}\lambda \text{iter }n}^S}{dz^S}$$  \hspace{1cm} (8)

$\mathbf{H}_{\text{RHS}}^S$ is calculated considering the initial undeformed configuration. It is possible to write the previous expression in the following explicit form:

$$H_{\text{RHS}}^S_{\text{step}\lambda \text{iter }n} = a^S + b^S x_{\text{step}\lambda \text{iter }n} + c^S u_{\text{step}\lambda \text{iter }n}$$  \hspace{1cm} (9)

where $a^S$, $b^S$ and $c^S$ are constant (this is a consequence of the made assumptions).

In order to obtain the vector $H_{\text{RHS}}^S_{\text{step}\lambda \text{iter }n}$ an assembling procedure, performed considering all wing segments, is required. The result of this procedure is the following equation:

$$H_{\text{RHS}}^S_{\text{step}\lambda \text{iter }n} = a + bx_{\text{step}\lambda \text{iter }n} + cu_{\text{step}\lambda \text{iter }n}$$  \hspace{1cm} (10)

where $a$, $b$ and $c$ are again constant quantities.

Using equations (8) and (10), the aerodynamic forces at load step $\lambda$ and iteration $n$ applied to the load points of the aerodynamic panels are:

$$L_{\text{step}\lambda \text{iter }n} = \lambda L_{\text{ref}} I \left[A^T \right]^{-1} (a + bx_{\text{step}\lambda \text{iter }n} + cu_{\text{step}\lambda \text{iter }n})$$  \hspace{1cm} (11)

In a compact form, the previous equation can be written as

$$L_{\text{step}\lambda \text{iter }n} = \lambda L_{\text{ref}} (\bar{a} + \bar{b} x_{\text{step}\lambda \text{iter }n} + \bar{c} u_{\text{step}\lambda \text{iter }n})$$  \hspace{1cm} (12)

where $\bar{a}$, $\bar{b}$ and $\bar{c}$ are constant matrices.

The aerodynamic loads of equation (12) are applied at the load points of the aerodynamic panels. They are transferred to the structural nodes with the following algorithm.

For all aerodynamic load points, the aerodynamic forces are extracted from equation (12). Then the triangular FEM element that includes the load point of the generic aerodynamic panel is found. The equivalent loads applied at the nodes of the triangular FEM element (which contains the load point) are obtained by using the area coordinates. Finally, an assembling procedure is required (a node in general connects more FEM
elements). Notice that some zero rows in correspondence of the rotational DOFs have to be added. The vector of the aerodynamic forces applied at the structural nodes is written as

\[ L_{\text{step} \lambda \text{iter} n} = \lambda L_{\text{ref}} (A + Bx_{\text{step} \lambda \text{iter} n} + Cu_{\text{step} \lambda \text{iter} n}) \]  

(13)

\[ A, B \] and \( C \) are constant matrices. Notice also that \( K_{\text{step} \lambda \text{Taero}} = -\lambda L_{\text{ref}} C \) is the aerodynamic tangent matrix. It is convenient to write equation (13) in the form

\[ L_{\text{step} \lambda \text{iter} n} = L_{\text{RHS}} + L_{\text{LHS}} \]  

(14)

where

\[ L_{\text{RHS}} = \lambda L_{\text{ref}} (A + Bx_{\text{step} \lambda \text{iter} n}) \]  

(15)

\[ L_{\text{LHS}} = \lambda L_{\text{ref}} Cu_{\text{step} \lambda \text{iter} n} = -K_{\text{step} \lambda \text{Taero}} u_{\text{step} \lambda \text{iter} n} \]  

(16)

The subscript LHS is used to point out that the term \( L_{\text{LHS}} \) will go on the left hand side of the equation iteratively solved in the Newton Raphson procedure (see equation (25) below).

![Figure 3. Wing surface S. Definition of the local coordinate system in the reference plane.](image)

V. Solution of the Nonlinear System Using the Newton Raphson Method

The wing is loaded by the aerodynamic loads and other loads (indicated with \( P_{\text{ext}} \)) such as the inertial loads. The used procedure (Newton Raphson) is the following (notice that the notation used here is slightly different than the notation used in the authors’ previous work):

- **Step # 1**
  
The reference aerodynamic pressure \( L_{\text{ref}} \) is calculated:

\[ L_{\text{ref}} = -\rho_{\infty} V_{\infty}^2 \frac{N_{\text{step}}}{N_{\text{step}}} \]  

(17)

Similar operation can be done for the external concentrated load. The reference amplitude \( P_{\text{ref}} \) of that loads will be

\[ P_{\text{ref}} = \frac{1}{N_{\text{step}}} \]  

(18)

- **Step # 2**
  
The applied non-aerodynamic loads are only step dependent and they are calculated by using the following expression:

\[ P_{\text{step} \lambda} = \lambda \cdot P_{\text{ref}} \cdot P_{\text{ext}} \]  

(19)

The aerodynamic loads are calculated:

\[ L_{\text{RHS}} = \lambda L_{\text{ref}} (A + Bx_{\text{step} \lambda \text{iter} n}) \]  

(20)

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where $\lambda$ is the load factor and it is equal to 1 for the first load step, 2 for the second load step and so forth. Notice that the vector $x^{\text{step } \lambda, \text{iter } n}$ is the vector $x$ calculated using the current coordinates of the structural nodes at the start of the iteration $n$. The internal forces $F^{\text{step } \lambda, \text{iter } n}_{\text{int}}$ are known from the previous iteration (if the very first iteration of the first load step is considered, there are no internal forces because the structure is stress-free). So the unbalanced loads $P^{\text{step } \lambda, \text{iter } n}_{\text{unb}}$ can be calculated:

$$P^{\text{step } \lambda, \text{iter } n}_{\text{unb}} = P^{\text{step } \lambda, \text{iter } n}_{\text{str}} + L^{\text{RHS}} - F^{\text{step } \lambda, \text{iter } n}_{\text{int}}$$  \hspace{1cm} (21)

**• Step # 3**

The structural tangent matrix $K^{\text{step } \lambda, \text{iter } n}_T$ is calculated by adding the elastic stiffness matrix $K^{\text{step } \lambda, \text{iter } n}_E$ (calculated considering the coordinates at the beginning of the $n^{\text{th}}$ iteration) and the geometric stiffness matrix $K^{\text{step } \lambda, \text{iter } n}_G$ (in the practice it is convenient to perform this operation at element level and then assemble the resulting matrix):

$$K^{\text{step } \lambda, \text{iter } n}_T = K^{\text{step } \lambda, \text{iter } n}_E + K^{\text{step } \lambda, \text{iter } n}_G$$  \hspace{1cm} (22)

The structural tangent matrix is updated at each iteration of the procedure. The aerodynamic tangent matrix is calculated:

$$K^{\text{step } \lambda}_T\text{aero} = -\lambda L^{\text{ref}} C$$  \hspace{1cm} (23)

Notice that the aerodynamic tangent matrix is only load step dependent. The reason is that the matrix $C$ is constant.

**• Step # 4**

The tangent matrix $K^{\text{step } \lambda, \text{iter } n}_{T\text{angent}}$ is built my adding the structural and aerodynamic tangent matrix in the following way:

$$K^{\text{step } \lambda, \text{iter } n}_{T\text{angent}} = K^{\text{step } \lambda, \text{iter } n}_T + K^{\text{step } \lambda, \text{iter } n}_{T\text{aero}}$$  \hspace{1cm} (24)

**• Step # 5**

The following linear system is solved and the displacement vector $u^{\text{step } \lambda, \text{iter } n}$ is found:

$$K^{\text{step } \lambda, \text{iter } n}_{T\text{angent}} \cdot u^{\text{step } \lambda, \text{iter } n} = P^{\text{step } \lambda, \text{iter } n}_{\text{unb}}$$  \hspace{1cm} (25)

**• Step # 6**

The coordinates are updated for the next iteration:

$$x^{\text{step } \lambda, \text{iter } (n+1)} = x^{\text{step } \lambda, \text{iter } n} + u_d^{\text{step } \lambda, \text{iter } n}$$  \hspace{1cm} (26)

$u_d^{\text{step } \lambda, \text{iter } n}$ is the vector which contains only the translational DOFs and it is obtained from the vector of displacements $u^{\text{step } \lambda, \text{iter } n}$ by eliminating the rows relative to the rotations. Notice that if the last iteration of the load step $\lambda$ has been performed, then the left hand side of the previous equation is $x^{\text{step } (\lambda+1), \text{iter } 1}$ instead of $x^{\text{step } \lambda, \text{iter } (n+1)}$.

**• Step # 7**

The rigid body motion is eliminated according to Levy-Gal procedure and the pure rotations and strains are found. With this quantities the internal forces are updated for the next iteration and, therefore, the vector $F^{\text{step } \lambda, \text{iter } (n+1)}_{\text{int}}$ is created (in the case in which the last iteration of load step $\lambda$ has been performed the term $F^{\text{step } \lambda, \text{iter } (n+1)}_{\text{int}}$ has to be replaced by $F^{\text{step } (\lambda+1), \text{iter } 1}_{\text{int}}$).

The procedure is repeated until the desired tolerance is reached.
VI. Approximated Aerodynamic Tangent Matrix Obtained from the Generalized Aerodynamic Tangent Matrix

The reduction of the system by using a known set of shape vectors (which can be for example the natural modes of the structure or a combination of natural modes and other assigned shapes) is a quite difficult operation if structures with geometrical nonlinearity are considered. For example, in the Joined Wing configuration the in-plane forces are important in the calculation of the geometric stiffness matrix. The used basis has to be able to capture such in-plane contributions in order to have a good approximation of the global behavior of the structure. The usual conventional approach, based on a selection of low frequency modes is usually not an efficient strategy. In fact, it is possible to show that a basis built by adopting this procedure is often not able to approximate the system and the results are poor, unless the basis is continuously updated, leading to expensive computation. Advanced procedures, such as the use of modes and modal derivatives,\(^{31}\) improve the approximation, but it is not enough for complex configurations. An alternative method, proposed in this paper, is to use the full order structural matrices to have a good representation of the in-plane forces (structural part), and a reduced aerodynamic matrix calculated, for example, by adopting a commercial software (ZAERO or other package). Actually, this procedure, shown in the paper for the steady case can be extended to the dynamic case making the representation of time domain simulations particularly effective when a commercial aerodynamic code is used for the generation of the generalized aerodynamic matrix in the frequency domain.

Consider a set \( \Psi \) of \( \mathbb{R} \) known shape vectors:

\[
\Psi = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \ldots & \Phi_N \end{bmatrix}
\]  

(27)

Let \( \mathbf{A}_0 \) be the generalized aerodynamic matrix (calculated for the steady case, so \( k = 0 \) where \( k \) is the reduced frequency) obtained by using a commercial code using the basis \( \Psi \). The matrix \( \mathbf{A}_0 \) has dimension \( \mathbb{R} \times \mathbb{R} \). The generalized aerodynamic tangent matrix \( \mathbf{A}_z \) is obtained simply by multiplying \( \mathbf{A}_0 \) by the fraction of the aerodynamic pressure related to the current load step \( \lambda \):

\[
\mathbf{A}_z = -\lambda \cdot \frac{1}{N_{\text{step}}} \frac{2 \rho \infty V^2}{N_{\text{step}}} \mathbf{A}_0 = +\lambda \cdot \frac{L_{\text{ref}}}{2} \mathbf{A}_0
\]  

(28)

Suppose we approximate the displacements of the generic iteration \( n \) of the generic load step \( \lambda \) by using the basis \( \Psi \) and the vector \( q^{\text{step} \lambda \text{iter } n} \) of generalized coordinates as follows:

\[
\mathbf{u}^{\text{step} \lambda \text{iter } n} \approx \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \ldots & \Phi_N \end{bmatrix} \begin{bmatrix} \mathbf{q}^{\text{step} \lambda \text{iter } n} \\
\mathbf{q}_1^{\text{step} \lambda \text{iter } n} \\
\mathbf{q}_2^{\text{step} \lambda \text{iter } n} \\
\vdots \\
\mathbf{q}_N^{\text{step} \lambda \text{iter } n} \end{bmatrix} = \mathbf{\Psi} q^{\text{step} \lambda \text{iter } n}
\]  

(29)

By applying the Least Square Method (LSM), it is possible to express the generalized coordinates as a function of the displacements. But the finite element degrees of freedom include nodal rotations which are usually not used in the interpolation of displacement and angle of attack over lifting surfaces for aeroelastic applications. The least squares approximation should focus on matching the modal approximation to the full order displaced shape of lifting surface panels. The relation displacements-generalized coordinates is represented by equation (29). Notice that \( \mathbf{u}^{\text{step} \lambda \text{iter } n} \) includes the rotations. By eliminating the rows which correspond to the rotations, the translational displacements \( \mathbf{u}_d^{\text{step} \lambda \text{iter } n} \) are found as

\[
\mathbf{u}_d^{\text{step} \lambda \text{iter } n} = \mathbf{\Psi}_d q^{\text{step} \lambda \text{iter } n}
\]  

(30)

where \( \mathbf{\Psi}_d \) contains only translational components of the \( \mathbb{R} \) shape vectors of the basis \( \Psi \). The dimension of \( \mathbf{\Psi}_d \) is then \( 3N_n \times \mathbb{R} \).

\[
\begin{align*}
\mathbf{u}_d^{\text{step} \lambda \text{iter } n} &= \mathbf{\Psi}_d q^{\text{step} \lambda \text{iter } n} \\
&= \mathbf{\Psi}_d^T \mathbf{u}_d^{\text{step} \lambda \text{iter } n} \\
&= \mathbf{\Psi}_d^T \mathbf{\Psi}_d \mathbf{q}^{\text{step} \lambda \text{iter } n}
\end{align*}
\]  

(31)

\[
\begin{align*}
q^{\text{step} \lambda \text{iter } n} &= \left( \mathbf{\Psi}_d^T \mathbf{\Psi}_d \right)^{-1} \mathbf{\Psi}_d^T \mathbf{u}_d^{\text{step} \lambda \text{iter } n} \\
&= \mathbf{T}_d \mathbf{u}_d^{\text{step} \lambda \text{iter } n}
\end{align*}
\]  

(32)

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where
\[ T_d = \left[ \Psi_d^T \Psi_d \right]^{-1} \Psi_d^T \]  \hspace{1cm} (33)

\( T_d \) is a matrix with dimension \( \mathcal{R} \times 3N_n \). Another possibility is to perform the LSM only on the component of the displacements perpendicular to the surfaces. Considering that the models analyzed in this paper are almost contained in the \( x - y \) plane, it is reasonable to assume that the displacement perpendicular to the surfaces is the translational displacement \( w \). So the LSM can be performed only on the translational displacement \( w \) of all nodes. If this approach is chosen, the matrix \( T_d \) has dimension \( \mathcal{R} \times N_n \) (instead of \( \mathcal{R} \times 3N_n \)), the matrix \( \Psi_d \) has dimension \( N_n \times \mathcal{R} \) (instead of \( 3N_n \times \mathcal{R} \)) and the vector \( u_d^{\text{step } \lambda \text{ iter } n} \) contains just \( N_n \) elements (instead of \( 3N_n \)).

It is preferable to work with the vector \( u^{\text{step } \lambda \text{ iter } n} \) instead of the vector \( u_d^{\text{step } \lambda \text{ iter } n} \). Therefore, \textit{columns of zeros} in correspondence of the finite element nodal rotations (and the displacements following definition is also made:

\[ q^{\text{step } \lambda \text{ iter } n} = Tu^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (34)

Now the generalized aerodynamic tangent matrix has to be converted in full order. This goal is achieved by adopting an energetic approach.

The \textit{generalized forces are} (the negative sign is a consequence of the used definition for \( \mathcal{A}_z \): see equation 28):

\[ Q^{\text{step } \lambda \text{ iter } n} = -\mathcal{A}_z q^{\text{step } \lambda \text{ iter } n} = -\lambda \cdot \frac{L_{\text{ref}}}{2} \mathcal{A}_0 q^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (35)

The work done by the generalized aerodynamic forces is:

\[ \delta W = \delta \left[ q^{\text{step } \lambda \text{ iter } n} \right]^T \cdot Q^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (36)

The work done by the equivalent aerodynamic forces \( \mathbf{L}_{LHS}^{\text{step } \lambda \text{ iter } n} \) applied on the structural mesh is:

\[ \delta W = \delta \left[ u^{\text{step } \lambda \text{ iter } n} \right]^T \cdot \mathbf{L}_{LHS}^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (37)

Using equation 34 and equating equation 39 and equation 37 it is possible to write:

\[ \delta \left[ u^{\text{step } \lambda \text{ iter } n} \right]^T \cdot \mathbf{L}_{LHS}^{\text{step } \lambda \text{ iter } n} = \delta \left[ q^{\text{step } \lambda \text{ iter } n} \right]^T \cdot Q^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (38)

From which it follows:

\[ \mathbf{L}_{LHS}^{\text{step } \lambda \text{ iter } n} = T^T \cdot Q^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (39)

Using equation 35 and 34

\[ \mathbf{L}_{LHS}^{\text{step } \lambda \text{ iter } n} = -\lambda \cdot \frac{L_{\text{ref}}}{2} T^T \mathcal{A}_0 q^{\text{step } \lambda \text{ iter } n} = -\lambda \cdot \frac{L_{\text{ref}}}{2} T^T \mathcal{A}_0 Tu^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (40)

To keep the formalism used when the aerodynamic tangent matrix was obtained using the vortex lattice formulation, we indicate the \textit{approximated aerodynamic tangent matrix} with the symbol \( K_{\text{TZ aero}}^{\text{step } \lambda} \). The following definition is also made:

\[ K_{\text{TZ aero}}^{\text{step } \lambda} = -\lambda \cdot L_{\text{ref}} C_Z \]  \hspace{1cm} (41)

\( C_Z \) is the approximated counterpart of the matrix \( C \) earlier defined.

Equation 40 is then rewritten as (see also the analogous equation 14 for the full order “exact” case)

\[ \mathbf{L}_{LHS}^{\text{step } \lambda \text{ iter } n} = -K_{\text{TZ aero}}^{\text{step } \lambda} u^{\text{step } \lambda \text{ iter } n} = +\lambda L_{\text{ref}} C_Z u^{\text{step } \lambda \text{ iter } n} \]  \hspace{1cm} (42)

Using equation 40 it can be deduced that

\[ C_Z = -\frac{1}{2} T^T \mathcal{A}_0 T \]  \hspace{1cm} (43)

\( K_{\text{TZ aero}}^{\text{step } \lambda} = -\lambda L_{\text{ref}} C_Z \) is the \textit{approximated aerodynamic tangent matrix} obtained by using the \textit{generalized aerodynamic matrix} \( \mathcal{A}_0 \) calculated with the aerodynamic commercial code.

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VII. Newton-Raphson Method Using the Approximated Aerodynamic Tangent Matrix

The procedure is very similar to the one used earlier. However, some steps of the procedure need to be revised. The deformation dependent aerodynamic effect is due to the total deformation from the initial stress-free configuration $x^{\text{start}}$. The equivalent aerodynamic forces that come from the approximated aerodynamic tangent matrix (see equation [12]) work with the vector of displacements and this vector includes the displacements and finite element nodal rotations. However, the fact that the aerodynamic forces do not depend on the rotations was taken into account when the LSM procedure was applied (the rows were eliminated, see equation [32]). Considering this fact, the initial stress-free undeformed configuration is augmented with zero rows in correspondence of the rotation DOFs. Let $x^{\text{start}}$ be the vector with the initial coordinates augmented by zero rows (similarly for $x^{\text{iter} \ n}$). The vector which contains all the aerodynamic forces is obtained by adding the contribution that comes from the undeformed configuration and the contribution that comes from the deformed configuration (measured from the initial shape). The last one can be split in two parts: in the first, the augmented initial coordinates and the augmented coordinates at the beginning of the iteration are present; the second part is just $L^{\text{iter} \ n}_{\text{LHS}} = + \lambda L_{\text{ref}} C_Z \mathbf{u}^{\text{iter} \ n} = - K_{\text{Zzero}} \lambda \mathbf{u}^{\text{iter} \ n}$. Thus, equation (20) of step #2 of the Newton Raphson procedure has to be replaced with the following expression:

$$L^{\text{iter} \ n}_{\text{RHS}} = \lambda L_{\text{ref}} (A + B x^{\text{start}}) + \lambda L_{\text{ref}} C_Z (x^{\text{iter} \ n} - x^{\text{start}})$$  \hspace{1cm} (44)

It has been numerically verified that when matrix $C_Z$ is replaced with the “exact” matrix $C$, equations (20) and (44) give the same results and the final deformed configuration is the same as well. Equation (23) of step #2 of the Newton Raphson procedure has to be replaced with the following (formally identical) expression:

$$K^{\text{iter} \ n}_{\text{Tangent}} = K^{\text{iter} \ n}_{T} + K^{\text{TZzero}}$$  \hspace{1cm} (45)

The other steps of the procedure are the same.

A. Calculation of the Generalized Aerodynamic Tangent Matrix by Using the Present Vortex Lattice Formulation

In general the procedure described here is tailored to work with generalized aerodynamic forces created by commercial linear unsteady aerodynamic codes. To check accuracy of the simulations based on generalized forces versus full order solutions in a consistent way, the same aerodynamic capability is used here to calculate both reduced order generalized forces and full order panel aerodynamic forces.

In order to test the described procedure, the matrix $A_0$ (in general obtained by adopting a commercial software) is generated by using the present vortex lattice formulation as described below. The matrix $-A_Z$ (the negative sign is a consequence of the used definition for $A_Z$: see equation (23)) assumes the meaning of generalized aerodynamic force matrix. The term $ij$ is the work done by the pressures due to motion in mode $j$ on the deformation on mode $i$. In order to calculate the work it is necessary to be able to calculate the angle of attack. This operation requires the use of splines. It is convenient, therefore, to use the previously obtained results and adopt all the matrices already derived. In particular, it was demonstrated (by using the vortex lattice formulation) the relation (repeated for convenience):

$$L^{\text{iter} \ n}_{\text{str}} = \lambda L_{\text{ref}} (A + B x^{\text{iter} \ n} + C \Phi)$$  \hspace{1cm} (46)

$L^{\text{iter} \ n}_{\text{str}}$ had the meaning of force vector (on the structural nodes) due to the aerodynamic loads. Now, it can be observed that a mode $\Phi_j$ has the meaning of displacements referred to the undeformed (stress-free) configuration (the coordinate vector of this configuration is indicated with $x^{\text{start}}$). Considering this fact, the aerodynamic forces on the structural nodes due to mode $j$ (indicated with $L^{j}_{\text{str}}$) are (see equation (45)):

$$L^{j}_{\text{str}} = \lambda L_{\text{ref}} (A + B x^{\text{start}} + C \Phi)$$  \hspace{1cm} (47)

The contribution due to the deformation of the structure is simply

$$L^{j}_{\text{def}} = \lambda L_{\text{ref}} C \Phi_j$$  \hspace{1cm} (48)
The work of the aerodynamic forces which depend upon the deformation (so the contribution of the initial aerodynamic forces present because in general the undeformed configuration has an angle of attack different than zero is not considered at this stage; it is considered on the RHS of the equation iteratively solved in the Newton Raphson procedure) is indicated with $W_{ij}^{\text{def}}$. It is due to the aerodynamic forces (depending on the mode $j$) that work through displacements of mode $i$ as follows:

$$W_{ij}^{\text{def}} = \Phi_i^T L_{i}^{\text{def}} = \lambda L_{\text{ref}} \Phi_i^T C \Phi_j$$  \hspace{1cm} (49)

The term $W_{ij}^{\text{def}}$ is also the element $ij$ of $-A_Z$. Remembering (see equation 28) that $A_Z = +\lambda \cdot \frac{L_{\text{ref}}}{2} A_0$, we get:

$$-A_Z_{ij} = -\lambda \cdot \frac{L_{\text{ref}}}{2} A_0_{ij} = W_{ij}^{\text{def}} = \lambda L_{\text{ref}} \Phi_i^T C \Phi_j$$  \hspace{1cm} (50)

Simplifying we finally get the expression for the generic element $ij$ of the matrix $A_0$:

$$A_0_{ij} = -2 \Phi_i^T C \Phi_j$$  \hspace{1cm} (51)

With this matrix, the full order approximated aerodynamic tangent matrix obtained from the generalized matrix can be calculated (see equation 43).

**VIII. Results**

Four different approaches have been investigated in this paper. In detail:

- **Method 1**
  The basis $\Psi$ is built by using natural modes. However the lumped mass matrix used to calculate the modes is modified by reducing the terms related to the $u$ and $v$ DOFs by 99%. With this reduction the modes have mainly the out-of-plane components and the basis is more indicated to approximate the out-of-plane displacement $w$. The Least Squares Method is performed only on the component $w$.

- **Method 2**
  The basis $\Psi$ is built by using natural modes but the mass matrix is not modified. The LSM is performed only on the component $w$.

- **Method 3**
  The basis $\Psi$ is built by using natural modes and the mass matrix is modified as in the Method 1. The LSM is performed on all the translational displacements $u$, $v$ and $w$.

- **Method 4**
  The basis $\Psi$ is built by using natural modes and the mass matrix is not modified as in the Method 2. The LSM is performed on all the translational displacements $u$, $v$ and $w$.

It should be noticed that the approximation with modes involves only the aerodynamic part of the aeroelastic problem. With this method, the structural part is exact and the very difficult problem of the simulation of a nonlinear reduced problem is avoided. Some success with modal reduction of structurally nonlinear problems has been reported in the literature, especially for cases of plate and shell structures with limited overall deformation. The joined-wing case is particularly difficult, with significant deformation, and with areas of stress concentration that affect in-plane action, and hence nonlinear structural behavior. The experience of the authors is that when both the aerodynamic and structural models are reduced with modes it is very difficult to capture well the in-plane stresses and a very high number of models is required and they have to be frequently updated during the iteration process. In addition, the modal derivatives are required to improve the accuracy of the results. With the present method, all these difficult issues are avoided and only the aerodynamic part is approximated.
A. Attar’s Delta Wing

The wing is shown in Figure 4. Attar’s original model is a delta wing. However, with the present formulation only trapezoidal wing segments are allowed. So the actual used model is slightly different (the tip has been modified to have a trapezoidal wing has shown in Figure 4).

In figure 5 the effect of the number of modes used in the basis $\Psi$ is analyzed. Two or three modes are sufficient to well approximate the aerodynamic tangent matrix and the correlation with the full order analysis is excellent. It will be shown that for more complex wing configurations more modes are required to have good accuracy. Tables 1 and 2 compare the tip displacement calculated at different load steps and the effect of the number of modes is reported as well. The number of digits used in the representation of the displacements (expressed in millimeters) is excessive (there is not much interest in fractions of millimeters), but it is reported to better show the comparison of the results obtained by using a different number of modes.

![Figure 4. Geometry of Attar’s delta wing.](image)

![Figure 5. Attar’s delta wing($\alpha = \frac{\pi}{180}$). Effect of the number of modes used in the approximation of the aerodynamic matrix (Method 1)](image)
<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Speed [m/s]</th>
<th>Full order</th>
<th>1 mode</th>
<th>2 modes</th>
<th>3 modes</th>
<th>4 modes</th>
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</table>

Table 1. Attar’s delta wing\((\alpha = \frac{\pi}{180})\). Displacement \(w[\text{mm}]\). Effect of the number of modes used in the approximation of the aerodynamic tangent matrix. The approximation is performed by using Method 1.

<table>
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<th>15 modes</th>
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Table 2. Attar’s delta wing\((\alpha = \frac{\pi}{180})\). Displacement \(w[\text{mm}]\). Effect of the number of modes used in the approximation of the aerodynamic tangent matrix. The approximation is performed by using Method 1.
B. Joined Wing

The geometry of the wing is shown in Figure 6. From Figure 7 and table 3 it is evident that with 15 modes the aerodynamics is well approximated and the correlation with the full order analysis is excellent. It has also to be pointed out that when the number of modes is too high (see Figure 8) the accuracy of the results is no longer good. This can be easily explained: in the LSM procedure when the number of shape functions is high the approximated shape is wavy and so the local angle of attack may be incorrect. If the angle of attack is incorrect, then the aerodynamic forces are not accurate and the error on the displacements may be large. In particular, when 360 modes are used, the final displacement is about 1.33 times the correct displacement (see table 4).

Table 5 shows that for an assigned number of modes Method 1 has better performances. This is a consequence of the fact that the configuration is almost contained in the plane $x - y$ and so the aerodynamic forces are mainly affected by the displacement $w$.

![Figure 6. Geometry of the Joined Wing model.](image)

![Figure 7. Joined Wing($\alpha = \frac{\pi}{180}$). Effect of the number of modes used in the approximation of the aerodynamic matrix (Method 1).](image)
Table 3. Joined Wing ($\alpha = \frac{4\pi}{180}$). Displacement $w[\text{dm}]$. Effect of the number of modes used in the approximation of the aerodynamic tangent matrix. The approximation is performed by using Method 1.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Speed [m/s]</th>
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<th>10 modes</th>
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Table 4. Joined Wing ($\alpha = \frac{4\pi}{180}$). Displacement $w[\text{dm}]$. Effect of the number of modes used in the approximation of the aerodynamic tangent matrix. The approximation is performed by using Method 1.

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Table 5. Joined Wing (\( \alpha = \frac{4\pi}{180} \)). Displacement \( w[\text{dm}] \). Comparison between different approximations of the aerodynamic tangent matrix. The approximation is performed by using 15 mode shapes.
IX. Conclusions

The goal of the present study was to combine the full order nonlinear finite element analysis of the structure with modally based generalized unsteady aerodynamic forces. The present work showed that for the steady case the suggested procedure works very well. Two-three modes are sufficient for classical configurations (such as Attar’s delta wing), while for more challenging cases (such as Joined-Wings), 15-20 modes are required.

The expression of the generalized coordinates as a function of the displacements is obtained by performing a Least Squares approximation and this operation has to be done on the translational components of the displacements only. However, in most wing configurations the displacements are mainly in the $z$ direction and if the Least Square Method is performed only on the component of the displacements $w$, then the results are more accurate. For the same reason, if the modes are artificially modified by reducing the mass entries of the mass matrix relative to the displacements $u$ and $v$ (a lumped mass matrix has been used) enhancing the out-of-plane component, the results are improved.

An excessive number of modes used in the definition of the basis adopted can deteriorate the approximation of the local angles of attack and, thus, the final accuracy of the approximated aerodynamic tangent matrix is poor and the results are not acceptable.

Unsteady cases will be described in a future paper.
References