Quasi-3D analysis of free vibration of anisotropic plates

Luciano Demasi *

Department of Aeronautics and Astronautics, Guggenheim Building, University of Washington, Seattle, WA 98195-2400, USA

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Abstract

The subject of the present work is multilayered finite elements that are able to furnish an accurate description of both strain and stress fields. The formulation of the finite elements is based upon Reissner’s mixed variational theorem (RMVT), which allows one to assume two independent fields for displacements and transverse stress variables. The resulting advanced finite elements can describe, a priori, the interlaminar continuous transverse shear and normal stress fields, and the so called $C_0^z$-requirements can be satisfied.

This paper is mainly concerned about the vibrations of multilayered plates. The present FEM formulation is validated by comparing the results with both the literature and the commercial code NASTRAN. To conduct the assessment, five challenging benchmarks with different boundary conditions and lamination schemes are considered. For each benchmark, the first four non-dimensional frequencies are calculated and a few modes are presented. It can be concluded that the present quasi-3D layer-wise formulation has a very good agreement with respect to the more computationally expensive three-dimensional FEM formulations. Moreover, the displacement and transverse stresses are computed a priori without any operation of post-processing. Therefore, the present results can be used as reference numbers for testing other new FEM models.

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Keywords: FEM; RMVT; Transverse stresses; Mixed formulation; Layer-wise models; NASTRAN; $C_0^z$ requirements

1. Introduction

Composite structures combine light weight, high stiffness, high structural efficiency and durability, and, therefore, have been used to build large portion of aerospace as well as automotive and ship vehicles.

As far as two-dimensional modeling of multilayered flat structures is concerned, there are a number of requirements that should be considered for an accurate description of their stress and strain fields.

First, anisotropic multilayered structures possess higher shear and normal flexibility than traditional isotropic one-layer ones. In fact, classical two-dimensional analyses of plates and shells based on Cauchy–Poisson–Kirchoff [1–3] assumptions (see Classical Lamination Theory (CLT) [4]), are inadequate to predict the global response of thick plates, Reissner–Mindlin [5–7], even though accounting for transverse shear deformations can lead to very inaccurate conclusions as far as local response of thick layered structures are concerned.

Second, the intrinsic discontinuity of the mechanical properties in the thickness direction of plate puts further difficulties on the two-dimensional modeling of layered structures. Exact three-dimensional solutions (see [8–11]) have shown that the displacements and the transverse stresses are $C_0^z$-continuous functions in the thickness $z$ direction (see Fig. 1 for the geometry and notations). In [12–14] these facts were referred to as $C_0^z$ requirements. Literature often marks these requirements as zig–zag form for the displacements $u = [u_x, u_y, u_z]^T$ and interlaminar continuity (i.e., equilibrium) for the transverse stresses $\sigma_n = [\sigma_{nx}, \sigma_{ny}, \sigma_{nz}]^T$. Many equivalent single-layer and layer-wise theories have been proposed.
and directed to overcome the limitation of CLT and FSDT and include, partially or completely, the above \( C_0^0 \) requirements. For a complete and detailed overview, see [15].

This paper assesses the vibrations of multilayered plates. The present FEM layer-wise mixed formulation is validated by comparing the results with both the literature and the commercial code NASTRAN. As previously discussed, thick composite plates require advanced theories/formulations, in order to capture all \( C_0^0 \) requirements. Therefore, the effectiveness of such quasi-3D layer-wise mixed formulation is demonstrated through five challenging benchmarks involving different boundary conditions and lamination schemes.

This paper is organized as follows:

- **Section 2**: \( C_0^0 \) requirements. The \( C_0^0 \) requirements are introduced.
- **Section 3**: Reissner’s mixed variational theorem (RMVT). The RMVT is introduced.
- **Section 4**: Layer-wise displacement and transverse stress assumptions. The displacement and stress field assumptions are detailed and illustrated.
- **Section 5**: RMVT: Equilibrium and compatibility equations. The equilibrium and compatibility equations, in the FEM approximation, are reported.
- **Section 6**: Free vibrations problem. The free vibrations problem is formulated using the present mixed formulation. Applying the static condensation technique, the problem is transformed into the standard form.
- **Section 7**: Results and discussion. The free vibrations problem is solved for several different cases. Comparison with the literature results is presented and five benchmarks are discussed.
- **Section 8**: Conclusion. The principal properties and results are discussed.

2. \( C_0^0 \) requirements

The layered construction (see Fig. 1) introduces complicating effects [12,13,16]. Transverse discontinuous mechanical properties, in fact, cause displacement fields (see Fig. 2(a)) in the thickness direction which can exhibit rapid changes and different slopes in correspondence to each layer interface. This is known as the zig–zag form of displacement field in the thickness direction. Although in-plane stresses (see Fig. 2(b)) can, in general, be discontinuous, equilibrium reasons demand continuous transverse stresses (see Figs. 2(c) and 3). This is often referred to, in the literature, as interlaminar continuity (IC) of transverse shear and normal stresses. Fig. 2 shows, from a qualitative point of view, what could be the scenario of displacement and transverse stresses distributions in a multilayered structure in exact solutions. Figs. 2 and 3 show that both displacements and transverse stresses, due to compatibility and equilibrium reasons, respectively, are \( C_0^0 \)-continuous functions in the thickness \( z \) direction. These conditions are referred to as \( C_0^0 \) requirements. The fulfillment of \( C_0^0 \) requirements is a crucial point in two-dimensional modeling of multilayered structures.

Summarizing, the \( C_0^0 \) requirements are:

- **Compatibility**: interlaminar continuity of the displacements.
- **Zig–zag**: interlaminar continuity of the displacements.
- **Equilibrium**: interlaminar continuity of the transverse stresses.

3. Reissner’s mixed variational theorem (RMVT)

In the previous section, it was shown that the fulfillment of \( C_0^0 \) requirements is very important in the multilayered structure models. Therefore, it is fundamental to have a tool which allows to model the transverse stresses and enforce the continuity through the thickness. Starting from the Hellinger–Reissner functional (see [17]), with a few transformations (partial Legendre transformation) it is not difficult to obtain the Reissner’s mixed variational theorem [18,19]. In the dynamic case, the RMVT can be expressed as

\[
\int_V \left( \delta \dot{e}_{pG} \sigma_{pl} + \delta \dot{e}_{nG} \sigma_{nM} + \delta \sigma_{nM} (\varepsilon_{nG} - \varepsilon_{nH}) \right) dV + \int_V \rho \ddot{u} \dot{u} dV = \delta L_e
\]

The subscripts “p” and “n” mean that the quantities are “plane” or “out-of-plane”; the subscript “G” means that the strains \( \varepsilon \) are calculated using the geometric relations (which express the strains as functions of the displacements); the subscript “H” underlines that the quantities are computed via Hooke’s law; and, finally,
the subscript “M” means that the stresses are those of the assumed model (the model is chosen such that it satisfies the IC requirement). \( \rho \) is the material density, \( u \) are the displacements, \( \varepsilon \) are the accelerations, \( L_e \) is the external virtual work and \( V \) denotes the three-dimensional multilayered body volume.

The relevant quantities involved in RMVT are explicitly reported:

- **Stresses and strains**
  \[
  \sigma_p = [\sigma_{xx} \sigma_{yy} \sigma_{xy}]^T, \quad \sigma_n = [\sigma_{zx} \sigma_{zy} \sigma_{zz}]^T
  \]
  \[
  \varepsilon_p = [\varepsilon_{xx} \varepsilon_{yy} \varepsilon_{xy}]^T, \quad \varepsilon_n = [\varepsilon_{zx} \varepsilon_{zy} \varepsilon_{zz}]^T
  \]

- **Hooke’s law**
  \[
  \sigma_p = C_{pp} \varepsilon_p + C_{pn} \varepsilon_n
  \]
  \[
  \sigma_n = C_{np} \varepsilon_p + C_{nn} \varepsilon_n
  \]

- **Mixed form of Hooke’s law**
  \[
  \sigma_p = C_{pp} \varepsilon_p + C_{pn} \varepsilon_n
  \]
  \[
  \varepsilon_n = C_{np} \varepsilon_p + C_{nn} \varepsilon_n
  \]

- **Relations between the two forms of Hooke’s law**
  \[
  C_{pp} = C_{pp} - C_{pn} (C_{nn})^{-1} \]
  \[
  C_{np} = -C_{pn} (C_{nn})^{-1}
  \]
  \[
  C_{nn} = (C_{nn})^{-1}
  \]

- **Strain–displacement relations**
  \[
  \varepsilon_p = D_p u
  \]
  \[
  \varepsilon_n = D_n u = (D_{n\Omega} + D_n) u
  \]
  where \( D_p, D_{n\Omega} \) and \( D_n \) are differential matrices.

### 4. Layer-wise displacement and transverse stress assumptions

#### 4.1. Displacement assumptions

The displacement components \( u_k^x, u_k^y \) and \( u_k^z \) of the \( k \)-layer (the total number of layers is indicated by \( N_l \)) are postulated, in the \( z \)-direction, according to the expansion
\[ u^k_x = F_t u^k_{x_1} + F_b u^k_{x_2} + F_s u^k_{x_3} = F_t u^k_{x_t}, \]
\[ u^k_y = F_t u^k_{y_1} + F_b u^k_{y_2} + F_s u^k_{y_3} = F_t u^k_{y_t}, \]
\[ u^k_z = F_t u^k_{z_1} + F_b u^k_{z_2} + F_s u^k_{z_3} = F_t u^k_{z_t}, \]
\[ \tau = t, b, r; \quad r = 2, \ldots, N - 1; \quad k = 1, \ldots, N_l \]

The subscripts \( t \) and \( b \) denote values related to the top and bottom layer surfaces, respectively. These two terms consist of the linear part of the expansion. The thickness functions \( F_t(\zeta_k) \) are defined at the \( k \)-layer level as
\[ F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \]
\[ F_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N \]
in which \( P_1 = P(\zeta_k) \) is the Legendre polynomial of \( j \)-order defined in the \( \zeta_k \) domain: \(-1 \leq \zeta_k \leq +1\). The chosen functions have the following properties:
\[ \zeta_k = \begin{cases} +1, & F_t = 1, \quad F_b = 0, \quad F_r = 0 \\ -1, & F_t = 0, \quad F_b = 1, \quad F_r = 0 \end{cases} \]

Eq. (7) can be written in a compact way
\[ u^k = N^k q^k \quad \tau = t, b, r; \quad r = 2, \ldots, N - 1; \quad k = 1, \ldots, N_l \]

where \( N \) denotes the number of nodes in the element, while \( q^k \) is the vector of the nodal displacement
\[ q^k = [q^k_{u_{11}}, q^k_{u_{12}}, q^k_{u_{13}}, q^k_{u_{21}}, q^k_{u_{22}}, q^k_{u_{23}}, q^k_{u_{31}}, q^k_{u_{32}}, q^k_{u_{33}}] \]

4.2. Transverse stress assumptions

The transverse stresses \( \sigma^k_{xi}, \sigma^k_{yi} \) and \( \sigma^k_{zi} \) are written as
\[ \sigma^k_{ni} = F_t \sigma^k_{ni}, \quad \tau = t, b, r; \]
\[ r = 2, \ldots, N - 1; \quad k = 1, \ldots, N_l \]

where
\[ \sigma^k_{ni} = N^k g^k_{ni} \quad (i = 1, 2, \ldots, N_n) \]

Notice that the previous assumptions for the displacement and stress fields satisfy a priori the \( C^2 \) requirements.

5. RMVT: Equilibrium and compatibility equations

Imposing the definition of virtual variations, Reissner’s mixed variational theorem leads to the following equilibrium and compatibility equations:
\[ \delta q^k_{ti}, K^{k_{xaij}} q^k_{ij} + K^{k_{xbij}} s^k_{ij} + M^{k_{xaij}} q^k_{ij} = P^k_{ti} \]
\[ \delta q^k_{ti}, K^{k_{xaij}} q^k_{ij} + K^{k_{xbij}} s^k_{ij} = 0 \]
where \( P^k_{ti} \) is the vector of external loads, \( q^k_{ij} \) and \( s^k_{ij} \) are the nodal displacements and transverse stress variables (layer \( k \)), and the matrices \( K^{k_{xaij}}, K^{k_{xbij}} \) and \( K^{k_{xaij}} (3 \times 3 \text{ matrices}) \) are the fundamental nuclei. \( M^{k_{xaij}} \) is the fundamental nucleus which allows one (by expanding the indexes) to write the consistent mass matrix. More details about the derivation of the matrices can be found in [16] and [20]. The expansion of the layer matrix from the correspondent \( 3 \times 3 \) fundamental nuclei is shown in Fig. 5. In Fig. 6 the assemblage from layer to multilayered level is shown.

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\(^1\) In this paper, only the case \( N_n = 9 \) will be considered.
6. Free vibrations problem

Expanding the layer matrices

\[ K^{\text{lsij}}_{uu}, K^{\text{lsij}}_{wo}, K^{\text{lsij}}_{aw}, K^{\text{lsij}}_{\sigma\sigma} \]  

at layer level and assembling the obtained matrices (see Figs. 5 and 6), the resulting matrices, at multilayered element level, are

\[ K_{uu}, K_{wo}, K_{aw}, K_{\sigma\sigma}, M \]  

Eq. (16), at multilayered element level, can be written as (notice that in the free vibrations problem the external loads are zero)

\[ \delta q : K_{uu} q + K_{wo} g + M \ddot{q} = 0 \]

\[ \delta g : K_{aw} q + K_{\sigma\sigma} g = 0 \]  

where \( q \) and \( g \) are the displacements and transverse stresses at multilayered element level. At this stage it is possible to apply the static condensation technique as follows:

\[ [K_{uu} - K_{wo}(K_{\sigma\sigma})^{-1}K_{aw}]q + M \ddot{q} = 0 \]  

By introducing

\[ K_{\text{mixed}}^{\text{LS}} = [K_{uu} - K_{wo}(K_{\sigma\sigma})^{-1}K_{aw}] \]

the governing equation written in terms of only displacement variables is obtained

\[ K_{\text{mixed}}^{\text{LS}}q + M \ddot{q} = 0 \]  

At structure level, one has

\[ K_{\text{mixed}}^{\text{LS}}q + M \ddot{q} = 0 \]  

which is the “classical” vibrations problem.

7. Results and discussion

7.1. Data of the treated problems

7.1.1. Materials

The materials that are used in this paper are:

- **MAT 1**
  \[ \frac{E_L}{E_T} = 40, \frac{G_{LT}}{E_T} = \frac{G_{LT}}{E_T} = 0.50, \frac{G_{TT}}{E_T} = 0.60, \]
  \[ u_{LT} = u_{LT} = v_{TT} = 0.25 \]

- **MAT 2**
  \[ \frac{E_L}{E_T} = 25, \frac{G_{LT}}{E_T} = \frac{G_{LT}}{E_T} = 0.50, \frac{G_{TT}}{E_T} = 0.20, \]
  \[ u_{LT} = u_{LT} = v_{TT} = 0.25 \]

7.1.2. Lamination schemes

The lamination schemes that are used in this paper are:

- **LS 1**: +0°/ +90°, \( h_1 = h_2 = h/2 \).
- **LS 2**: +0°/ +45°, \( h_1 = h_2 = h/2 \).
7.1.3. Mesh

In this paper, two different FEM elements are compared. The first FEM element is LM4 (see above) which is a two-dimensional element (Q9). Therefore, a two-dimensional mesh is required when such element is used. The second FEM element is a three-dimensional NASTRAN element (EXA with 8 nodes). Therefore, a three-dimensional mesh is required when such element is used.

7.1.4. Boundary conditions

The boundary conditions are explained in Fig. 7.

7.1.5. Description of the benchmarks

Five different benchmarks are proposed. The details are the following:

- **MESH 1**: $4 \times 4$ (LM4 only).
- **MESH 2**: $5 \times 5$ (LM4 only).
- **MESH 3**: $16 \times 16 \times 4$ (NASTRAN only).
- **MESH 4**: $24 \times 24 \times 6$ (NASTRAN only).

For example, MESH 4 has 24 EXA elements in the x direction, 24 EXA elements in the y direction and 6 EXA elements in the z direction. In all cases, the plates are square plates with edge $a$ and thickness $h$.

Table 1

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<th>NASTRAN (3)</th>
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Table 3

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Table 4

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<tr>
<td>LM4 (MESH 1)</td>
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</table>

- **LS 3**: $-45^\circ/+45^\circ$, $h_1 = h_2 = h/2$.
- **LS 4**: $-30^\circ/+45^\circ$, $h_1 = h_2 = h/2$.
- **LS 5**: $-30^\circ/+30^\circ$, $h_1 = h_2 = h/2$.

Fig. 7. Boundary conditions.
Table 5
Benchmark 4. Comparison LM4 vs NASTRAN

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Table 6
Benchmark 5. Comparison LM4 vs NASTRAN

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- **Benchmark 1**
  Square plate (edge \( a \) and thickness \( h \)). Boundary conditions: CCCC, CFCF, CCFF. Material MAT 2. Lamination scheme LS 1.

- **Benchmark 2**
  Square plate (edge \( a \) and thickness \( h \)). Boundary conditions: CCCC, CFCF, CCFF. Material MAT 2. Lamination scheme LS 2.

- **Benchmark 3**
  Square plate (edge \( a \) and thickness \( h \)). Boundary conditions: CCCC, CFCF, CCFF. Material MAT 2. Lamination scheme LS 3.

- **Benchmark 4**
  Square plate (edge \( a \) and thickness \( h \)). Boundary conditions: CCCC, CFCF, CCFF. Material MAT 2. Lamination scheme LS 4.

7.2. Comparison with some results available in the literature

The present FEM formulation is compared with the analytical results found by Carrera [21]. As shown in Table 1, the correlation for both thick and thin plates is very good.
7.3. Proposed benchmarks: comparison with NASTRAN

The present FEM formulation is compared with NASTRAN. Different mesh and boundary conditions are analyzed. In all cases (see Tables 2–6) the correlation between LM4 and NASTRAN is excellent. It is possible to observe that the frequency parameters change significantly if the boundary conditions are changed. The modes, for the benchmark 5, are plotted\(^2\) in Figs. 8–13. Finally, a comparison with the degree of freedom (DOF) used in both LM4 and NASTRAN is reported in Table 7. As can be seen, LM4 can capture the frequencies and modes with good accuracy using a significantly less number of DOF. Moreover, LM4 is a two-

\(^2\) In the LM4 case, the modes are plotted at the top surface of the plate.
dimensional formulation, while NASTRAN (in this paper) is a three-dimensional formulation.

8. Conclusion

The present quasi-three-dimensional layer-wise formulation has shown very good correlation with the analytical results available in the literature for both thin and thick plates.

Five challenging benchmarks with different lamination schemes and boundary conditions have been proposed. In all cases, the present FEM model has shown very good agreement with the commercial code NASTRAN. Considering the results, it can be concluded that RMVT is a very powerful tool: the $C^0_z$ requirements are satisfied a priori and no post processing operations are required in order to calculate the transverse stresses. Therefore, the benchmarks reported here can be used as reference numbers for future FEM formulations.

References